CALCULATION OF THE INTEGRATED HEAT-TRANSFER COEFFICIENTS FOR FREE CONVECTION IN CLOSED AXIALLY SYMMETRIC VESSELS

Yu. A. Kirichenko, A. A. Motornaya,
and P. S. Chernyakov

An approximate method of calculating the integrated heat-transfer coefficients for free convection based on certain assumptions justified by experiment [1] is described. The equations of free convection are analyzed in dimensionless form.

1. Consider an axially symmetric vessel filled with incompressible, viscous liquid, having an initial temperature T_0, the thermal flux density q being specified on the surface of the vessel for $t > 0$. The process of free convection is described by the following system of equations [2]:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \operatorname{grad})\mathbf{v} = -\operatorname{grad} p + P\Delta\mathbf{v} - GP^2(T - \langle T\rangle)\mathbf{i}$$
$$\operatorname{div}\mathbf{v} = 0, \quad \frac{\partial T}{\partial t} + (\mathbf{v}, \operatorname{grad})T = \Delta T \tag{1.1}$$

and the following initial and boundary conditions for S:

$$\mathbf{v}|_{t=0} = 0, \quad \mathbf{v}|_{x\in S} = 0, \quad T|_{t=0} 0, \quad \frac{\partial T}{\partial n}\Big|_{x\in S} = q$$
$$T = \frac{T_1\lambda}{q_m l_0}, \quad T_0 = \frac{T_{10}\lambda}{q_m l_0}, \quad q = \frac{q_1}{q_m}, \quad \mathbf{v} = \frac{\mathbf{u} l_0}{a}$$
$$t = \frac{a t_1}{l_0^2}, \quad p = \frac{p_1 l_0^2}{\rho a^2}, \quad G = \frac{g\beta q_m l_0^4}{\lambda \nu a}, \quad P = \frac{\nu}{a}$$

Here T is the dimensionless temperature, T_1 is the temperature of the liquid, λ is the thermal conductivity of the liquid, q_m is the maximum value of the density q_1, l_0 is the characteristic linear dimension, T_0 is the dimensionless initial temperature of the liquid, T_{10} is the true initial temperature of the liquid, $\langle T\rangle$ is the dimensionless volume-average temperature, **v** is the dimensionless velocity of the liquid, t is the dimensionless time, t_1 is the true time, a is the thermal diffusivity of the liquid, p is the dimensionless pressure, p_1 is the true pressure, ρ is the density of the liquid, **i** is a unit vector directed along the acceleration of the earth's gravity, G is the Grashof number, ν is the kinematic viscosity, β is the coefficient of volume expansion, g is the acceleration of the earth's gravity, P is the Prandtl number, **n** is the normal to the surface S defining the region Ω.

Having written down the equation of thermal balance and integrated this for the initial condition $\langle T\rangle|_{t=0} = T_0$, we obtain

$$\langle T\rangle = T_0 + \frac{Qt}{V}, \quad Q = \int_S q\,dS, \quad V = \frac{V_1}{l_0^3} \tag{1.2}$$

Here V_1 is the volume of the region occupied by the liquid.

Let us suppose that the following conditions are satisfied:

1) The flow of liquid is laminar, quasi-stationary, and axially symmetrical or plane;

2) the Grashof number is much greater than unity, and the Prandtl number is of the order of unity;

Khar'kov. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 140–143, September–October, 1970. Original article submitted April 28, 1969.

3) the whole region occupied by the liquid may be divided into a thermal boundary layer δ thick, a dynamic boundary layer, and the core of the liquid;

4) the flow of liquid in the core is ideal;

5) the temperature of the core equals the volume-average temperature $\langle T \rangle$;

6) the thickness of the thermal boundary layer equals the thickness of the dynamic boundary layer;

7) the thickness of the thermal boundary layer is constant [1];

8) the thermophysical properties are independent of temperature.

Let us consider the basis of assumptions (1)-(8). Assumption (2) does not seriously restrict the problem, since for a whole series of liquids (for example, cryogenic liquids such as liquid oxygen, nitrogen, hydrogen, etc., as well as alcohols and water) the Prandtl number is of order of unity, while the condition $G \gg 1$ corresponds to developed convection. Assumption (6) is a consequence of (2); for $P \sim 1$ the thermal and dynamic boundary layers coincide [3]. Assumptions (5), (7), and (partially) (1) correspond to the experimental results of [1], which indicate that for $t \sim 10^{-3}$ convective flow passes into the quasistationary mode, and for a large part of the volume the thickness of the thermal boundary layer is constant, while the temperature of the core of the liquid differs very little from the volume-average value. The theoretical model thus constitutes a certain idealization of the experimental results.

We seek the temperature field T and the velocity field v in the following form:

$$T = \langle T \rangle + \tau(x, y) \qquad \mathbf{v} = \mathbf{v}(x, y) \tag{1.3}$$

Substituting (1.3) into (1.1) and (1.2) and using the propositions of boundary-layer theory [3], we obtain

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = P\frac{\partial^2 u}{\partial y^2} + GP^2\tau\Phi(x) \tag{1.4}$$

$$u\frac{\partial \tau}{\partial x} + v\frac{\partial \tau}{\partial y} = \frac{\partial^2 \tau}{\partial y^2} - \gamma \quad \left(\gamma = \frac{Q}{V}\right) \tag{1.5}$$

$$\frac{\partial}{\partial x}(uR_0(x)) + R_0(x)\frac{\partial v}{\partial y} = 0$$

$$x = \frac{x_1}{l_0}, \quad y = \frac{y_1}{l_0}, \quad u = v_{x1} \quad v = v_{y1} \quad \Phi = i_x \tag{1.6}$$

Here x_1, y_1 is a coordinate system connected to the surface S; the origin of coordinates is the point of intersection of the symmetry axis with the lower part of the surface of the vessel; $R_0(x)$ is the radius of curvature of the vessel cross section.

The boundary conditions for (1.4)-(1.6) are

$$\begin{gathered} u|_{y=0} = 0, \quad v|_{y=0} = 0, \quad u|_{y=\delta} = f, \quad \frac{\partial u}{\partial y}\bigg|_{y=\delta} = 0 \quad \left(\delta = \frac{h}{l_0}\right) \\ \tau|_{y=\delta} = 0, \quad \frac{\partial \tau}{\partial y}\bigg|_{y=0} = -q(x), \quad \frac{\partial \tau}{\partial y}\bigg|_{y=\delta} = 0 \quad \left(f = \frac{f_1 l_0}{a}\right) \end{gathered} \tag{1.7}$$

In this we assume that heat passes from the wall of the vessel into the boundary layer, and that heat and mass transfer then take place from the boundary layer into the core.

Here h is the thickness of the thermal boundary layer, f_1 is the longitudinal component of the velocity of the core at the interface with the boundary layer ($y = \delta$).

In order to solve the boundary problem (1.4)-(1.7), we use the integral-relationship method [3]. We seek the temperature profile in the following form:

$$\tau = \tau_0 + \tau_1\frac{y}{\delta} + \tau_2\left(\frac{y}{\delta}\right)^2 + \tau_3\left(\frac{y}{\delta}\right)^3 + \tau_4\left(\frac{y}{\delta}\right)^4 \tag{1.8}$$

The coefficients τ_0, τ_1, τ_2, τ_3, τ_4 are found from the conditions

$$\begin{gathered} \tau|_{y=\delta} = 0, \quad \frac{\partial \tau}{\partial y}\bigg|_{y=\delta} = 0, \quad \frac{\partial^2 \tau}{\partial y^2}\bigg|_{y=\delta} = \gamma \\ \frac{\partial \tau}{\partial y}\bigg|_{y=0} = -q, \quad \frac{\partial^2 \tau}{\partial y^2}\bigg|_{y=0} = \gamma \end{gathered} \tag{1.9}$$

Substituting (1.8) into (1.9) we obtain the systems of equations

$$\begin{gathered}\tau_0 + \tau_1 + \tau_3 + \tau_4 = 0, \ \tau_2 = 0.5\gamma\delta^2 \\ \tau_1 + 2\tau_2 + 3\tau_3 + 4\tau_4 = 0, \ \ 2\tau_2 + 6\tau_3 + 12\tau_4 = \gamma\delta^2\end{gathered} \tag{1.10}$$

Solving (1.10) we obtain

$$\begin{gathered}\tau_0 = 0.5\delta q, \ \tau_1 = -q\delta, \ \ \tau_2 = 0.5\gamma\delta^2 \\ \tau_3 = \delta\,(q - \gamma\delta), \ \tau_4 = 0.5\delta\,(\gamma\delta - q)\end{gathered} \tag{1.11}$$

The profile of the longitudinal component u we seek in the form

$$u = A_0 \frac{y}{\delta} + f_1\left(\frac{y}{\delta}\right)^2 + A_2\left(\frac{y}{\delta}\right)^3 \tag{1.12}$$

We find the coefficients A_0, f_1, A_2 from the following conditions:

$$u\,|_{y=\delta} = f, \ u\,|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0, \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = -\frac{GP\Phi\, q\delta}{2} \tag{1.13}$$

Substituting (1.12) into (1.13) and solving the resultant system of equations, we obtain

$$\begin{gathered}A_0 = {}^3/_2 f - {}^1/_2 f_1, \ \ f_1 = -{}^1/_4 GP\delta^3 q\Phi, \ \ A_2 = -{}^1/_2\,(f + f_1) \\ u = \frac{1}{2}\left[f\left(\frac{3y}{\delta} - \frac{y^3}{\delta^3}\right) + f_1\left(\frac{2y^2}{\delta^2} - \frac{y^3}{\delta^3} - \frac{y}{\delta}\right)\right]\end{gathered} \tag{1.14}$$

After integrating (1.4) and (1.5) and using (1.6) and (1.7), we obtain respectively

$$\begin{gathered}\frac{d}{dx}\int_0^\delta u^2 R_0(x)\,dy - f\,\frac{d}{dx}\int_0^\delta u R_0(x)\,dy = -P R_0(x)\left.\frac{\partial u}{\partial y}\right|_{y=0} + GP^2\Phi\, R_0(x)\int_0^\delta \tau dy \\ \frac{d}{dx}\int_0^\delta u\tau R_0(x)\,dy = q\,(x)\,(1 - \gamma\delta)\,R_0(x)\end{gathered} \tag{1.15}$$

Substituting (1.8), (1.11), (1.14), into (1.15) and integrating the resultant equations with respect to x, we obtain a system of two algebraical equations for δ and f.

The dimensionless integrated heat-transfer coefficient N is calculated from

$$N = \frac{\alpha\lambda}{q_m l_0} = \frac{2}{\delta}$$

Here α is the integrated heat-transfer coefficient.

Let us consider some particular cases of the foregoing problem.

2. Let us consider free convection in a torus for which a constant thermal flux density is specified on the surface (Fig. 1). In this case

$$R_0(x) = l + \sqrt{l^2 + \sin^2 x}, \ \Phi(x) = \sin x, \ q(x) = 1, \ \gamma = 2$$

The equation for δ thus takes the form

$$\begin{gathered}-{}^{39}/_{70} k_0 F_1^2 + F_1 F_2\,(6Pk_4 - Ra\delta^5 k_9) + F_2^2\delta^2\,(k_{10} Ra^2\delta^8 - k_{11} R_n P\delta^3 + 2k_3 RaP\delta^4) = 0 \\ F_1 = 2k_4 - 4k_4\delta - Ra\delta^5 k_7 - Ra\delta^6 k_8 \\ F_2 = s_0 k_6 + 2s_1\delta k_0, \ k_a = \sqrt{l^2 + 1} - l \\ k_1 = \sqrt{l^2 + 1} + 2l^3 - {}^2/_3\,(l^2 + 1)^{/_2} \\ k_2 = 0.5\sqrt{l^2 + 1} - 0.5\, l^2\, ln\,[(1 + \sqrt{l^2 + 1})/l] \\ k_3 = 1.5l + 0.5\,(1 + l^2)\ \text{arc sin}\,(l^2 + 1)^{-1/2} \\ k_4 = {}^1/_2\, l\pi + l^{-1}\sqrt{l^2 + 1}\, E\,((l^2 + 1)^{-1/2}) \\ k_5 = {}^1/_3\,(l^2 + 1)^{3/2} + {}^1/_2 l - {}^1/_3 l^3 \\ k_6 = {}^1/_2\sqrt{l^2 + 1} + l + 0.5\, l^2 \ln\,[(1 + \sqrt{l^2 + 1})/l] \\ k_7 = 0.25 s_2\,(k_2 + k_6), \ \ k_5 = 2s_3\,(k_2 + k_6) \\ s_0 = -1.15535715, \ \ s_1 = 0.02202381, \ \ \ s_2 = 0.0684525, \ \ s_3 = -0.17857\cdot 10^{-2}\end{gathered} \tag{2.1}$$

Here $E(z)$ is an elliptic integral of the first kind and R_a is the Rayleigh number.

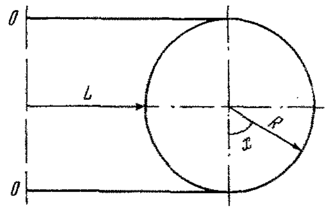

Fig. 1

The value of the longitudinal velocity component of the core at the interface with the boundary layer is calculated from the formula

$$f = \frac{F_1}{F_2}$$

A solution of Eq. (2.1) on an electronic computer (M-20), using the Muller method [4], for $R_a = 10^7$–10^{11} gave eight complex and four real roots, of which one was positive and smaller than unity. This root was taken as the thickness of the thermal boundary layer. An analysis of the results showed than δ and N were independent of the Prandtl number. After analysis of the results, we obtained the equations

$$N = (0.661 + 0.0357l)\,(R_a)^{0.196}$$
$$\delta = \frac{2}{0.661 + 0.0357\,l}\,(R_a)^{-0.196} \qquad (2.2)$$

Here $R_a = GP$.

As defining dimension in Eq. (2.2) we taken the radius of the torus. The maximum error in calculating N by Eq. (2.2) is 8%.

3. In the same way as in Section 2 we also considered the problem of free convection in a sphere having a constant thermal flux density specified on its surface. As a result of calculations analogous to those just carried out in the range of Rayleigh numbers $10^7 \le R_a \le 10^{11}$ we obtained the following equations:

$$N = 1.044\,(R_a)^{0.193}, \quad \delta = 1.92\,(R_a)^{-0.193} \qquad (3.1)$$

As defining parameter in (3.1) we take the radius of the sphere. The following are the values of the longitudinal components of the velocity of the core f at its interface with the boundary layer

$GP =$	10^7	10^8	10^9	10^{11}
$f =$	324.5	855.1	2176	13360

The values of the Nusselt number calculated from (3.1) for $GP = 10^9$, 10^{11} will respectively be N = 57.5, 140; according to experimental data [1] they are respectively N = 54, 123.

This recommends Eq. (3.1) for use in calculating the integrated heat-transfer coefficient for free convection in a sphere over the range of Rayleigh numbers $10^7 \le R_a \le 10^{11}$.

4. In the same way as in 2 we also considered free convection in an infinite horizontal cylinder with a constant thermal flux density specified on its surface. For $R_a = 10^7$–10^{11} we analogously obtained the following equations:

$$N = 0.711\,(R_a)^{0.1943}, \quad \delta = 2.81\,(R_a)^{-0.1943} \qquad (4.1)$$

As defining parameter in Eqs. (4.1) we took the radius of the cylinder.

The satisfactory agreement between theory and experiment for one of the cases considered (the sphere) suggests the general validity of the model employed and recommends the proposed method of calculation for the range $R_a = 10^7$–10^{11} when calculating the integrated heat-transfer coefficients and temperature distribution in the boundary layer for free convection in all axially symmetric vessels.

LITERATURE CITED

1. Yu. A. Kirichenko and V. N. Shchelkunov, "Heat transfer in free convection in a spherical vessel," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 5 (1968).
2. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media [in Russian], Gostekhizdat, Moscow (1954).
3. S. S. Kutateladze, Fundamentals of the Theory of Heat Transfer [in Russian], Mashgiz, Moscow–Leningrad (1957).
4. I. S. Berezin and N. P. Zhidkov, Computing Methods [in Russian], Vol. 2, Fizmatgiz, Moscow (1952).